Pergamon

Int. J. Solids Structures Vol. 33, No. 15, pp. 2243–2259, 1996

0020-7683/96 $15.00 + .00

0020-7683(95)00054-2

THREE-DIMENSIONAL STRESS ANALYSIS OF FREE-EDGE EFFECTS IN A SIMPLE COMPOSITE CROSS-PLY LAMINATE

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(*Received April* 1994; *in revised form* 27 *February* 1995)

Abstract—A new model has been developed to analyze the free-edge effect in a symmetric cross-ply laminate. This new technique involves a three-dimensional finite element analysis that divulges the stress field at the free edge of a simple composite plate of finite dimensions. Three-dimensional 20-node quadratic brick elements were used to simulate the composite laminate. The cross-ply laminate is subjected to uniaxial tensile strain. The advantages of the "Slice Model Setup" are demonstrated in terms of computer time and memory. The slice model has been developed to simulate the actual case, and anticipated stress singularities have therefore not been induced artificially by using singular elements. The state of stress is determined and the results are favorably compared to the results of others found in the literature. The effect of thickness and width of a simple composite plate on the magnitude of stress at the free edge is also investigated. The purpose of the slice model is to create a reasonable finite element model whose state of stress can undergo failure analysis and whose elements can permit the use of property degradation techniques. Copyright © 1996 Published by Elsevier Science Ltd.

1. INTRODUCTION

It has been postulated that out-of-plane stresses at the free edge of a composite laminate plate may lead to delamination, especially when the specimen has been subjected to fatigue loading. An early analysis of the interlaminar shear stresses in a composite laminate has been done by Puppo and Evensen (1970). They showed that interlaminar shear stresses reach their maximum magnitudes at the free edges. Pipes and Pagano (1970) subsequently analyzed the interlaminar behavior of a simple composite plate, subjected to uniform axial strain. They utilized classical linear theory of elasticity to reveal the presence of significant stresses, both normal and shear, at the edges, between plies of composite plates.

The existence of these stress singularities at the free edge is an important factor in creating delamination failure of the composite plate. To accurately examine the stress state on the free edge of composite laminates, numerous people have investigated the problem. For this purpose, different techniques were applied by various authors. A common approach to this problem was an analytical closed form solution. This method has been employed by Pagano (1970, 1974, 1978), Sierakowski and Ebcioglu (1970), Chen and Sih (1971), Hein and Erdogan (1971), Tang (1976), Wang and Crossman (1978), Wang and Choi (1983a,b), Kassapoglou and Lagace (1987), and Becker (1993), among others. Finite element analysis was also a very popular technique, utilized by Rybicki (1971), Isakson and Levy (1971), Pian and Mau (1972), Wang and Crossman (1977), Spilker (1980), and Chang *et al.* (1988). Other techniques used have been finite difference methods, Pipes (1972), and Altus *et al.* (1980), boundary layer theory, Tang and Levy (1975), perturbation methods, Hsu and Herakovich (1977a,b), Galerkin method, Wang and Dickson (1978), and experimental methods, Pipes and Daniel (1971), and Harris and Orringer (1978). Rodini and Eisenmann (1978) investigated the problem by experimental methods as well as by a closed form theoretical approach. Rohwer (1985) formulated a three-dimensional slice, of dimensions Δx, but simplified it into a quasi-three-dimensional analysis. Kress (1992) later used a quasi-three-dimensional finite element model to examine the width effect at the free edge.

Because of major difficulties in numerical and closed form techniques, such as computer time, satisfying all boundary conditions, existence of mathematical singularities in the stress

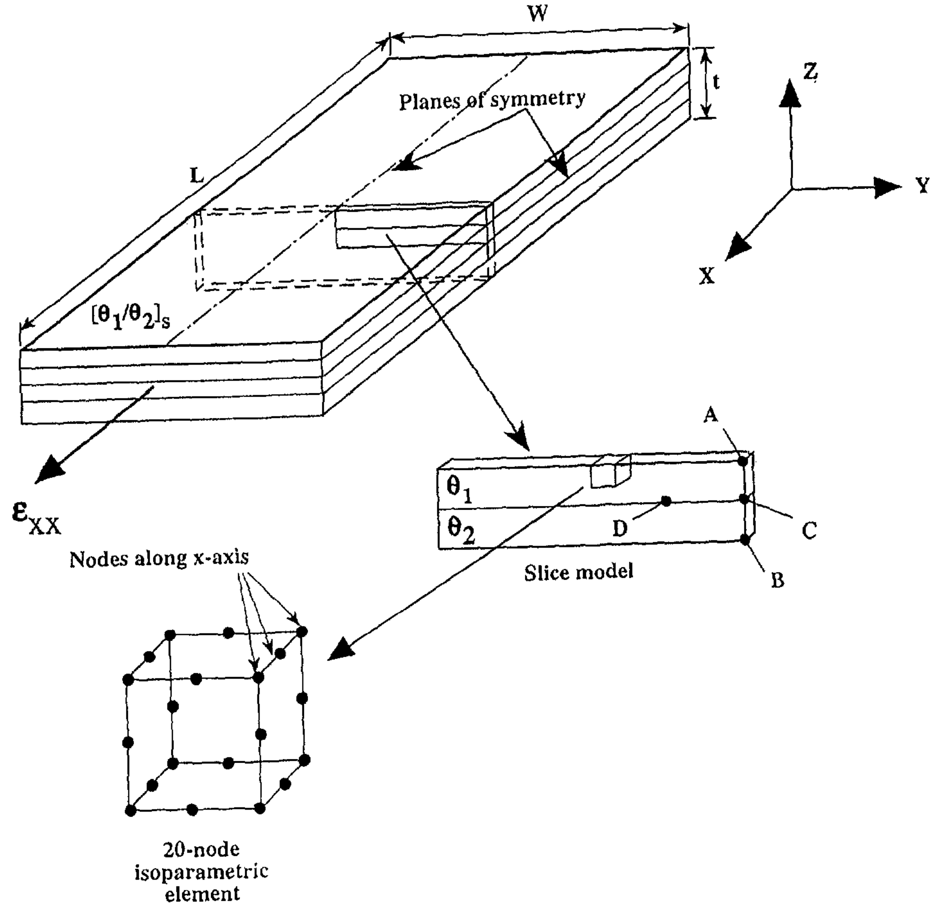

Fig. 1. Simple composite plate, slice model, showing element type used.

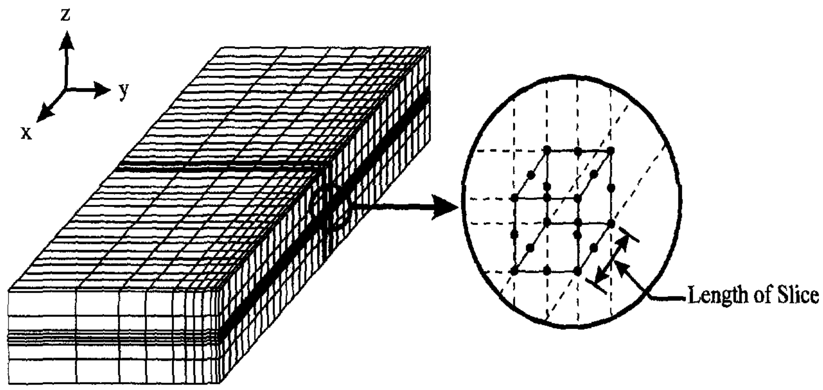

Fig. 2. Meshing of one quarter of simple composite plate, showing length of slice model.

Table 1. Model characteristics and runtime (values for 3-D models have been estimated)

	Computer runtime	Number of elements	Bandwidth
Slice model coarse	3 min 11 s	240	400
Slice model medium	19 min 54 s	826	1369
Slice model fine	24 min 8 s	856	1434
Fully 3-D model fine	1.22×10^7 years	8,070,000	8,877,000
3-D model coarse (aspect ratio = 1/10th)	813 years	234,000	360,000
3-D model coarse (max. extended elements)	297 days	23,400	36,000

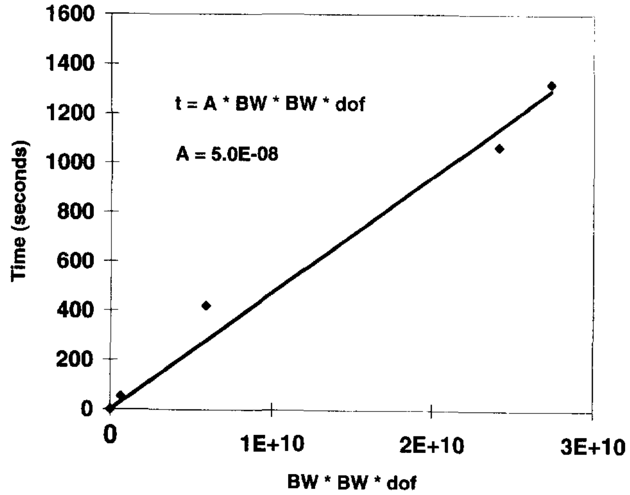

Fig. 3. Determination of constant A. Data points are the results of runtimes from the slice models. (Solution times only.)

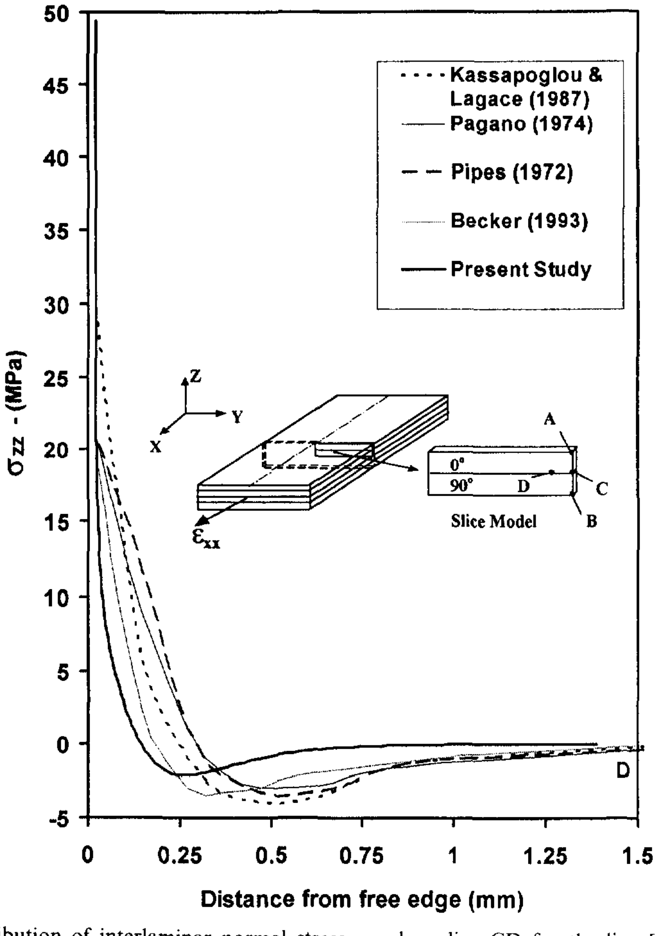

Fig. 4. Distribution of interlaminar normal stress σ_{zz} along line CD for the line [0/90]$_s$ simple composite plate. Comparison with results obtained by Kassapoglou and Lagace (1987), Pagano (1974), Pipes (1972), and Becker (1993).

field, etc., approximate techniques were used by Pipes and Pagano (1970). They assumed that normal stress was constant inside a composite plate, while varying linearly near the edges. Their results now seem to be coarse with respect to the results that have been obtained by other techniques. Kassapoglou and Lagace (1986) have contributed significantly by introducing an analytical solution based upon the principle of minimum complementary energy. Their technique is very efficient when determining the interlaminar stress state of thick (up to one hundred plies) simple composite plates. Rose and Herakovich (1991), later extended the work of Kassapoglou and Lagace to adapt it for all angle-ply laminates. Despite the quantity of work using closed form techniques, it is important to realize that all closed form solutions use approximation assumptions and are thus not exact solutions.

Raju *et al.* (1980), reviewed previous works and showed discrepancies in the results obtained by different authors. They also investigated the reliability of finite element displacement formulations in the analysis of the free edge problem.

Almost all investigators believe that mathematically, there is a stress singularity at the free edges, between different layers of composite plates, although Wang and Crossman (1977), Spilker and Chou (1980), and Spilker (1980) showed that interlaminar stress distributions converge to large but finite magnitudes. Spilker and Chou are, however, the only ones who seem convinced that no singularities exist. Zwiers *et al.* (1982), Wang and Choi (1982a,b), and Wilkins *et al.* (1982), have undertaken various independent studies to prove the presence of stress singularities near the edges. The form, strength, and power of the singularities have yet to be accurately resolved. Whitcomb *et al.* (1982) investigated the discrepancies in the results obtained by different authors and showed the reliability of the finite element approach, as well as indicating the presence of stress singularities. They concluded that the finite magnitudes of stress obtained by various authors were the result of using a symmetric stress tensor at the free edge, where an unsymmetric stress tensor is present. Bauld *et al.* (1985) arrived at the same conclusions, using both finite difference and finite element methods.

There remain many areas that have not yet been fully investigated, and by considering the importance of the three-dimensional stress state at the free edge of a composite plate, it is clear that the problem is open to further investigation. An important consideration is to provide a model that can provide an accurate stress state as well as allowing for a complete failure analysis of the simple composite plate. A complete three-dimensional failure analysis of the simple composite plate under a uniform ramp load and/or fatigue loading which includes property degradation features can be undertaken with the results provided by the slice model. Property degradation is possible since the slice model has the ability to change the material properties in all directions independent of one another. The use of 20-node quadratic isotropic elements are suitable for this purpose. These elements allow for a modification of material properties that is essential for continued failure analysis once an element has failed. The slice model is an attempt to solve these problems and allow for a failure analysis to be performed.

2. DESCRIPTION OF THE PROBLEM

Consider a simple composite plate which has width W, length L, half-width b, thickness t and number of plies n (Fig. 1). The composite laminate plate is subjected to a uniaxial tension by application of constant longitudinal strain, ε_{xx}. The composite plate is made up of various plies of 0° and 90° ply orientation to form cross-ply laminates. Each ply is assumed to be transversely isotropic material, and the laminates are always symmetric with respect to the midplane. The following is a list of the material properties for a typical high modulus graphite-epoxy unidirectional composite system, from Pipes and Pagano (1970), and the physical properties used during the finite element analysis.

$$E_{11} = 20 \times 10^6 \text{ psi} = 137.9 \text{ GPa}$$

$$E_{22} = E_{33} = 2.1 \times 10^6 \text{ psi} = 14.5 \text{ GPa}$$

$$G_{12} = G_{13} = G_{23} = 0.85 \times 10^6 \text{ psi} = 5.86 \text{ GPa}$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

$$\varepsilon_{xx} = 0.01 \text{ (applied strain)}$$

$$t_{\text{ply}} = 0.134 \text{ mm} = 5.276 \times 10^{-3} \text{ in}$$

$$h = 12.7\,\text{mm} = 0.5\,\text{in (unless otherwise stated)}$$

$$n = 4\ (\text{for } [0/90]_s \text{ ply lay-up})$$

$$L_{slice} = 0.0156\,\text{mm} = 6.142 \times 10^{-4}\,\text{in (see Fig. 1)}$$

where the 1, 2, and 3 are on-axis material directions, and x, y and z are the loading directions.

3. FINITE ELEMENT ANALYSIS

In order to study the edge effects, delamination, out-of-plane buckling and stacking sequence effects of the composite plate, a fully three-dimensional model must be utilized. To achieve high accuracy, the isoparametric quadratic 20-node brick element was used during finite element analysis. Singular elements near the free edge were not utilized for this reason, and since it is desired to simulate the problem without deliberately inducing the expected stress singularities. Although an unsymmetric stress field is present at the interface corner, the use of unsymmetric stress tensors is not desirable, because it has been shown by Raju *et al.* (1980), that σ_{zz} is forced to be positive (incorrectly) at that location if σ_{yy} and σ_{xy} are set to be zero.

3.1. *The fully three-dimensional model*

The problem of the simple composite plate has been studied in the past by a number of sources, many of whom were interested in the interlaminar edge effects. The conventional approach has been to simulate the specimen with a full size three-dimensional model. The advantages are clear: an accurate representation of the specimen at all locations, an analysis that produces all six stress and strain components, and the possibility to apply a suitable failure analysis and property degradation technique.

There are however limitations and drawbacks to this method of stress analysis related to computer restrictions. First and most important, the computer has a fixed amount of random memory allocated which limits the size of the model. This memory restriction implies a maximum number of nodes and elements that can be used in the finite element model. The consequence for the simple composite plate is that too few elements can be placed between the plies and along the free edge of the model. The stress analysis results in an inaccurate representation of the stress field at the free edges. Another major disadvantage of the fully three-dimensional model is the problem of excessive computer runtime. Therefore, to resolve these difficulties, a new finite element model was desired.

3.2. *Introduction of the slice model*

In order to alleviate the difficulties encountered by the fully three-dimensional model, the "Slice Model" has been developed. The slice model drastically reduces the number of elements required by the fully three-dimensional model to simulate a simple composite plate, while at the same time retaining the use of 20-node quadratic brick elements.

The slice model was created by cutting out a thin slice from the composite plate (Fig. 1). The thin slice was modeled with a length of one element (simulated length of slice, $L_{slice} = 0.0156$ mm), which drastically reduced the number of elements required to model the simple composite plate. A uniaxial tensile strain was applied to the nodes on one side of the elements while the nodes on the opposite side of the elements were restrained in the longitudinal (axial) direction (Fig. 1). In order to closely approximate the actual situation, all nodes along the same longitudinal (axial, i.e. x-direction) line were constrained to move the same amount in the y- and z-directions. The highest stress gradients were expected to occur near the free edge (line AB, Fig. 1) and between the plies (line CD, Fig. 1). By considering the symmetry of the $[0/90]_s$ and the $[90/0]_s$ simple composite plate with respect to the xy- and zx-planes, one quarter of the specimen is modeled (Fig. 1).

Three variations of the slice model were created to develop models that could be used to demonstrate the importance of the meshing technique and element selection when obtaining accurate stress magnitudes via finite element analysis. The three models are: the coarse, medium and fine meshed slice models. At first, the "coarse model" with 240 brick elements and 1838 nodes was created. The coarse model has few elements and is used mainly to find the high stress regions. The coarse meshing is employed by quick estimates of critical stress regions, when the magnitudes of the stresses are not important. The coarse model had 20 nodes in 0.1 mm along the free edge near the ply interface. Thereafter, a medium mesh size was modeled which increased the number of elements, especially in the critical region (between the layers along the free edge). The medium model provides more accurate stress results compared to the coarse model, but a significant increase in runtime occurs. The number of elements was increased to 826, and in the critical region between the plies at the free edge, there were 20 nodes in 0.01 mm. Finally, the fine slice model was selected, which again increased the number of elements in the critical region to 20 nodes in 0.002 mm. A total of 6225 nodes and 856 elements were used in the fine slice model. In the fine model and the medium model, elements were concentrated near the critical region whereas the meshing remained coarse at non-critical regions. For all three cases, the aspect ratio of the elements near the free edges never exceeds a 2 : 1 ratio. For accurate results of the magnitude of stress singularities, the fine slice model is recommended.

3.3. *Comparison of the slice model and the fully three-dimensional model*

The reason for using the slice model instead of simulating a fully three-dimensional composite plate involves computer limitations. The number of elements required to accurately model a fully three-dimensional composite plate is much higher than the slice model, since the slice model requires only one element along the length while the full model requires many more (see Fig. 2). The full model would be severely limited by computer capacity. The number of elements near the critical free edge region would be dramatically reduced and the runtime increased, with results that are far less accurate than the slice model. The resulting magnitudes of stresses between the plies at the free edge of the laminate would not be as accurate as desired, far less than those of the slice model.

This method, in effect, reduces the size of the model as well as the runtime dramatically. When the magnitudes of the high stress regions are not essential but rather the locations and forms are important, the coarse slice model would be used. The model can also more accurately determine the magnitude of the stress near the expected singularities with the use of the fine slice model. The slice model setup can also be extended to solve 8-layer laminate plates, in which case three different interfaces must be considered. Accurate results can be obtained, while an increase in runtime would be involved.

The computer utilized during finite element analysis is a HP 730 series workstation. The computer specifications are as follows : 64 megabytes of random memory, 99 megahertz clock speed, and a MIPs integer of 76.

The descriptions of each slice model and its computer runtime required to solve by finite element analysis have been listed in Table 1. There are significant differences in computer runtime between the various slice models. The runtime varies from under 5 minutes (core runtime of the coarse meshed slice model) to over 90 minutes (fine model 8 plies). To achieve the degree of accuracy obtained by the fine meshed slice model, a large number of elements and a significant runtime are required.

The computer runtime for the slice model is much lower than the time required to solve a fully three-dimensional model. The runtimes required by full 3-D models have been estimated by using the following equation : $Time = A * (Bandwidth)^2 * dof$. In this equation, dof represents the number of degrees of freedom of the model, while A is a constant which must be determined. The runtime was plotted versus the $(Bandwidth)^2 * dof$ factor for the slice model results in Fig. 3. The value of A was determined as the slope of the best fit line of Fig. 3. The bandwidth, for the fully three-dimensional models, is estimated because the computer used during finite element analysis does not have enough memory to obtain a solution. The runtimes of these models, Table 1, are then determined from the value of A determined in Fig. 3, and the estimated bandwidths.

The fully three-dimensional model represents a 152.4 mm simple composite plate, the shortest specimen that can be tested experimentally. To achieve the same type of meshing as the slice model, the number of elements required for a full three-dimensional model is far higher. To acquire the accuracy of the slice model, an element every 0.0156 mm is required. The 1/10th aspect ratio model, with one element every 0.156 mm, assumes a higher aspect ratio (and thus less accurate results), but with the same type of meshing. The final model tabulated represents an estimation of the maximum number of elements required to simulate coarse meshing of a fully three-dimensional model with optimum use of extended elements.

In the absence of an *h*-version finite element method, Gupta *et al.* (1991), to reduce the number of elements, such as the slice model, the computer runtime would be far too great (in the order of years). Unless a technique similar to the slice model, or a model with elements that are much too long (with an aspect ratio that is too high) is utilized, the fully three-dimensional model cannot be solved with an accuracy that is even close to that of the slice model. It can therefore be concluded that to achieve the desired results with the type of elements desired, the slice model setup is a good solution. The slice model is capable of finding the balance between quality of results and minimal computer runtime.

3.4. *Advantages of three-dimensional finite element analysis*

It is also important to note that the three-dimensional finite element analysis of the simple composite plate has a number of advantages when compared to other methods used to study this problem. These other techniques, such as quasi-three-dimensional analysis and closed form solutions, can also obtain the stress state of the free edge of the simple composite plate. The slice model, however, can closely determine the state of stress (all six stresses), and has an acceptable runtime.

It is important to realize the major advantage of the three-dimensional stress analysis. However, the class of problems that can be analyzed by the slice model are limited to the cases of uniaxial tension and uniaxial compression for a limited number of plies.

4. RESULTS

The results of the stress analysis for both cross-ply models, $[0/90]_s$ and $[90/0]_s$, and their comparison to results obtained from literature are presented here. Due to the nature of the cross-ply laminate, both σ_{xy} and σ_{zx} are zero throughout the problem, as expected. This behavior agrees with results obtained by classical laminate theory. The stress analysis was performed with SDRC I-DEAS finite element software.†

4.1. *Comparison with previously developed models*

The stress analysis of the simple composite plate has been studied on many previous occasions, and many references exist to check the effectiveness and accuracy of the slice model: Pipes (1972), Pagano (1974), Kassapoglou and Lagace (1987), and Becker (1993). The results of the stress analysis for the $[0/90]_s$ "fine-meshed" slice model as compared to other works is presented here. In Fig. 4, the interlaminar normal stress σ_{zz} is plotted along the $0°$–$90°$ ply interface. The results obtained by Pagano (1974) by approximate closed-form solution, Pipes (1972) by numerical solution, Kassapoglou and Lagace (1987) by energy-variational closed form solution, and Becker (1993) by closed-form solution, are included in Fig. 4. The graph shows that the results obtained with the use of the slice model compare very favorably with those obtained by other authors. The general shape of the stress variation matches the results of the other authors, while the maximum stress at the free edge displays a larger peak stress. The latter indicates the ability of the slice model to properly detect possible stress singularities.

4.2. *Slice model results (mesh dependence and convergence)*

The simple composite plate was studied to show the importance of element creation and selection during analysis. In Figs 5(a–d) the interlaminar normal (σ_{zz}) and shear (σ_{yz}) stresses along the free edge of the laminate, along line AB (see Fig. 1), are sketched for both cross-ply laminates $[0/90]_s$ and $[90/0]_s$. Each of the four graphs contains three curves:

† SDRC, I-DEAS (1990). Integrated Design Engineering Software. User Manual, Structural Dynamic Research Corporation.

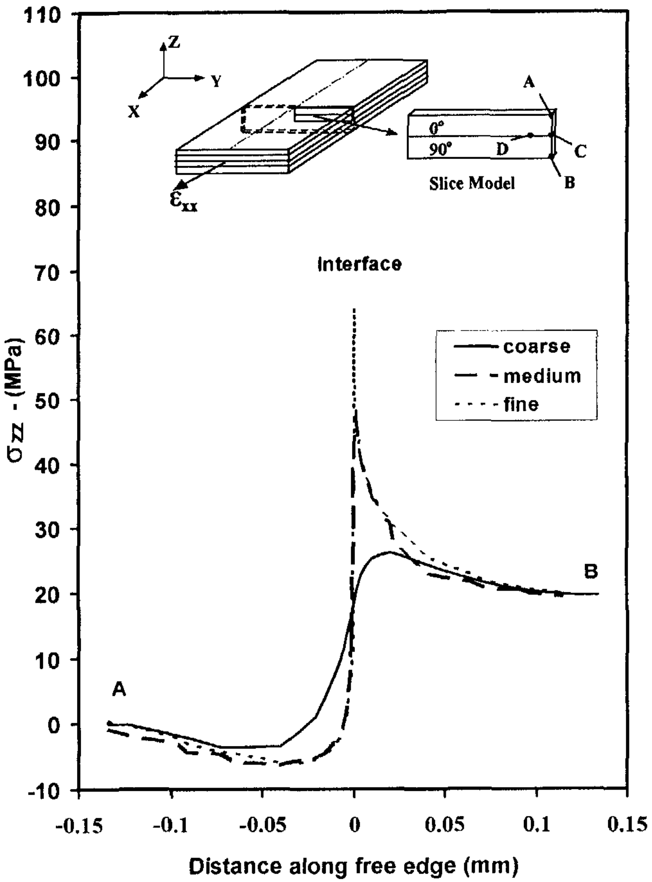

Fig. 5a. Distribution of interlaminar normal stress σ_{zz} along line AB for the $[0/90]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

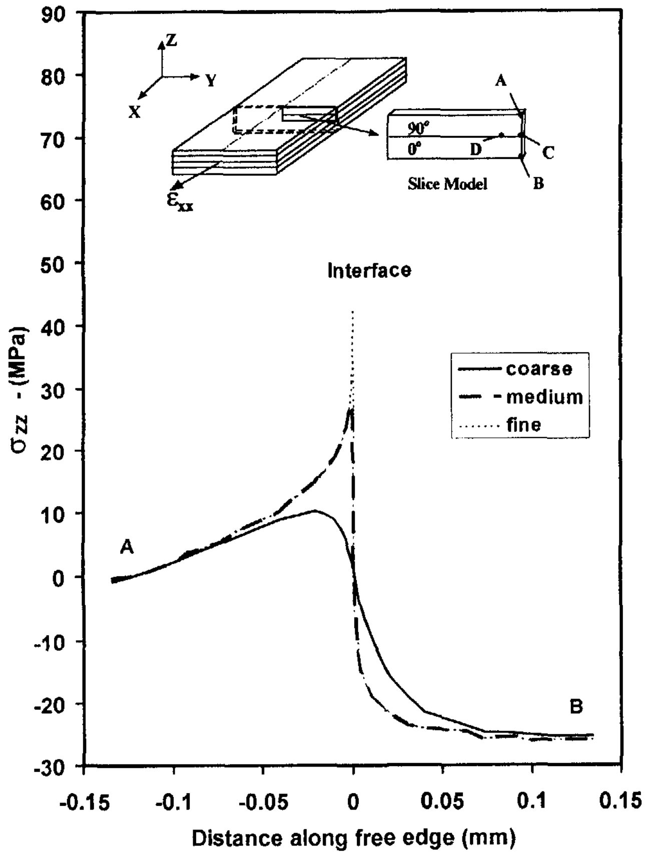

Fig. 5b. Distribution of interlaminar normal stress σ_{zz} along line AB for the $[90/0]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

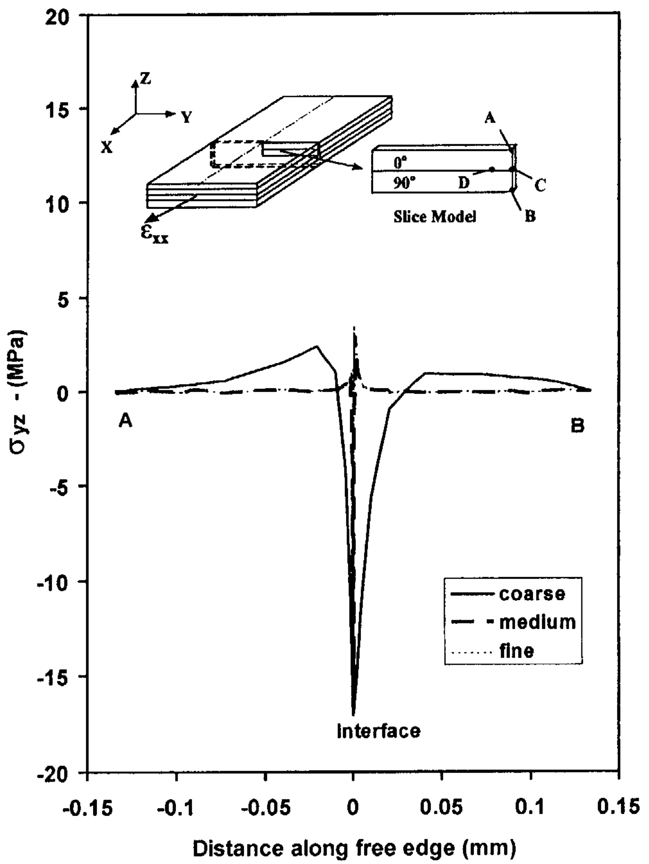

Fig. 5c. Distribution of interlaminar shear stress σ_{yz} along line AB for the $[0/90]_s$ simple composite plate. Results of fine, medium and coarse meshing of the slice model have been plotted.

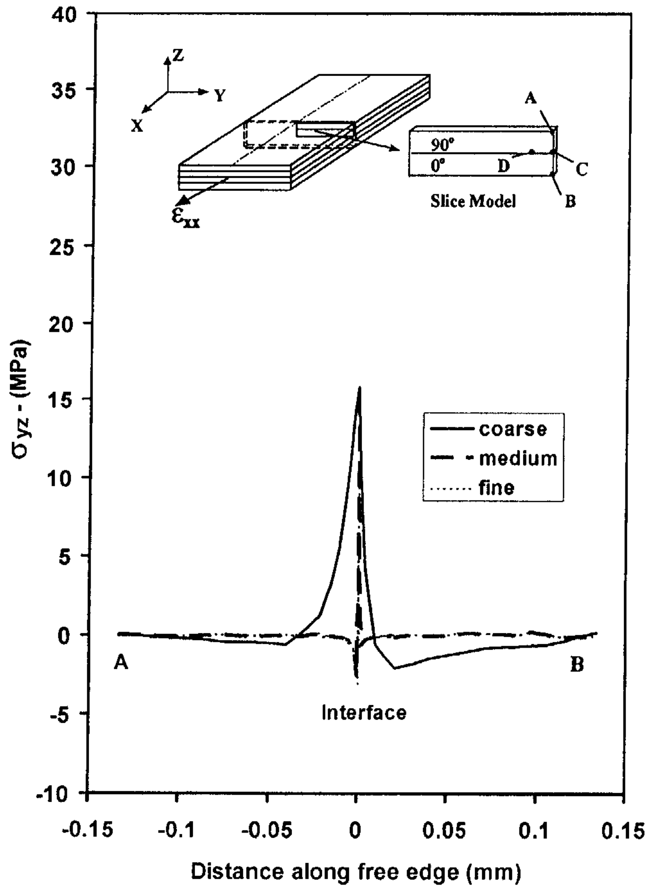

Fig. 5d. Distribution of interlaminar shear stress σ_{yz} along line AB for the $[90/0]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

coarse, medium, and fine models. The results show that in all four figures, the magnitude of the stress is dependent on the type of meshing employed.

In Fig. 5(a), the interlaminar normal stress σ_{zz} along the free edge of the $[0/90]_s$ simple composite plate is plotted for fine, medium and coarse meshing of the slice model. It is observed that the stress is dependent upon the meshing utilized. Convergence is approached between the medium and fine meshes, since the fine mesh shows little further changes from the medium mesh. It is also shown that σ_{zz} is zero at the top of the laminate (position A). In Fig. 5(b), a further analysis shows that the interlaminar normal stress between the plies is smaller for the $[90/0]_s$ laminate (29 MPa for the fine model) when compared to $[0/90]_s$ [49 MPa for the fine curve of Fig. 5(a)]. In Figs 5(c,d) it can be seen that the magnitude of interlaminar shear stress at the edge between the plies is not drastically affected by the mesh type utilized. It must also be mentioned that although the signs are opposite, the magnitudes of σ_{yz} are similar for the $[0/90]_s$ and $[90/0]_s$ configurations are relatively close. Once again the weaknesses of the coarsely meshed model can be observed. The high stress region occurring between the plies is accurately illustrated by the medium and fine models, while the coarse model shows a wider area of high interlaminar shear stress. The expected value of zero interlaminar shearing stress at the free edge is not precisely obtained since the elements at the free edge could not be meshed small enough. This is a limitation of this type of finite element modeling.

The stresses between the plies, along line CD (see Fig. 1) are displayed for the $[0/90]_s$ and $[90/0]_s$ cases in Figs 6(a–d). In Fig. 6(a), the distribution of interlaminar normal stress σ_{zz} along line CD for the $[0/90]_s$ simple composite plate with results of the fine, medium, and coarse meshing of the slice model have been plotted. The possible stress singularity is properly indicated by all three models, and the general shape of the stress variation is in concurrence for all cases. The magnitude of the stress at the edge is however very much dependent on the type of meshing. The employment of the fine meshed slice model is therefore important when an accurate value of the magnitude of the normal interlaminar stress at the free edge between the plies is desired.

The weaknesses of the coarse model are much more evident for all the interlaminar normal stress σ_{zz} along line CD for the $[90/0]_s$ laminate [Fig. 6(b)]. The magnitudes of the stress clearly suggest a stress singularity for the medium and fine meshed cases, but the coarse model does not indicate a stress singularity. An example of this can be shown when comparing to the work of Wang and Crossman (1977), who concluded that: "σ_{zz} for $[0/90]_s$ displays a sharp rise toward the free-edge; the sharpness of the rise suggests a possible stress singularity. On the other hand, the behavior of σ_{zz} for $[90/0]_s$ shows an entirely different nature; no stress singularity is indicated". The conclusions obtained from the coarse slice models would be identical to those reached by Wang and Crossman, however when the results of the medium and fine models are studied, stress singularities seem to be indicated for both cross-ply laminates.

It can be concluded that the magnitude of the interlaminar normal stress (σ_{zz}) is higher for the $[0/90]_s$ simple composite plate than for the $[90/0]_s$ case. As stated previously, the interlaminar shear stress σ_{yz} for both cross-ply cases are of the same magnitudes. The $[90/0]_s$ composite laminate is more resistant to delamination than the $[0/90]_s$ case because the normal stress is of smaller magnitude. It may therefore be an advantage to use a $[90/0]_s$ cross-ply laminate if delamination is a concern, as concluded by Pagano and Pipes (1971).

Once again the stress distribution of the interlaminar shear stress has also been investigated. Figures 6(c,d) represent the interlaminar shear stress (σ_{yz}) along line CD for the $[0/90]_s$ and the $[90/0]_s$ composite laminate plates respectively; fine, medium, and coarse results are once again illustrated. The coarse model seems to show evidence of a stress singularity for both cases while the medium and fine models reach a peak magnitude before reducing in magnitude. The magnitude of σ_{yz} reduces significantly with finer meshing. This seems to indicate that with a sufficiently fine meshing and enough elements to model the slice, the free edge condition of σ_{yz} equal to zero could be shown. The results obtained by the fine and medium models compare very well both in general shape and appearance as well as magnitude with the results obtained by Pipes (1972), Pagano (1974), Kassapoglou and Lagace (1987), and Becker (1993).

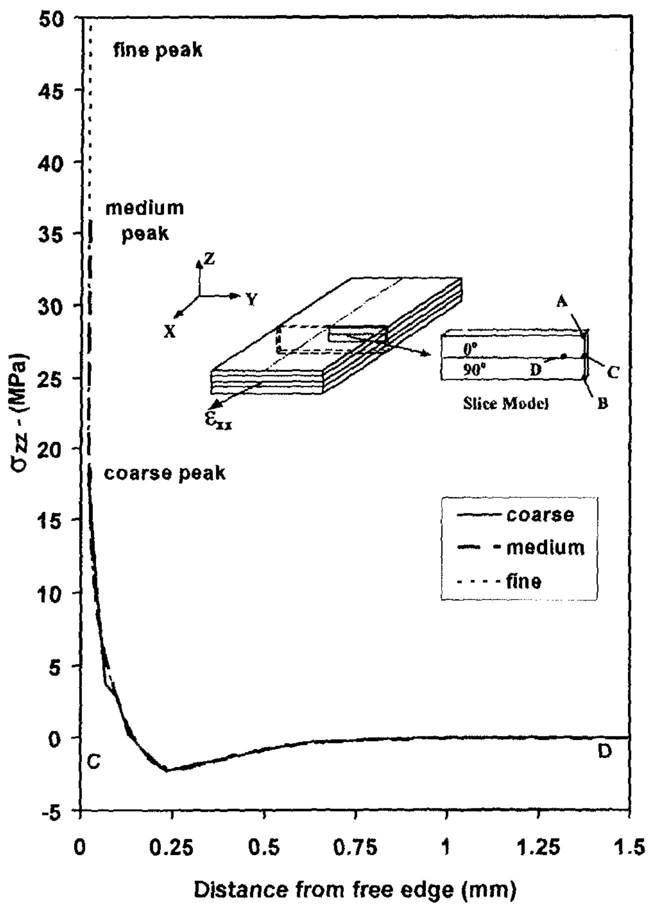

Fig. 6a. Distribution of interlaminar normal stress σ_{zz} along line CD for the $[0/90]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

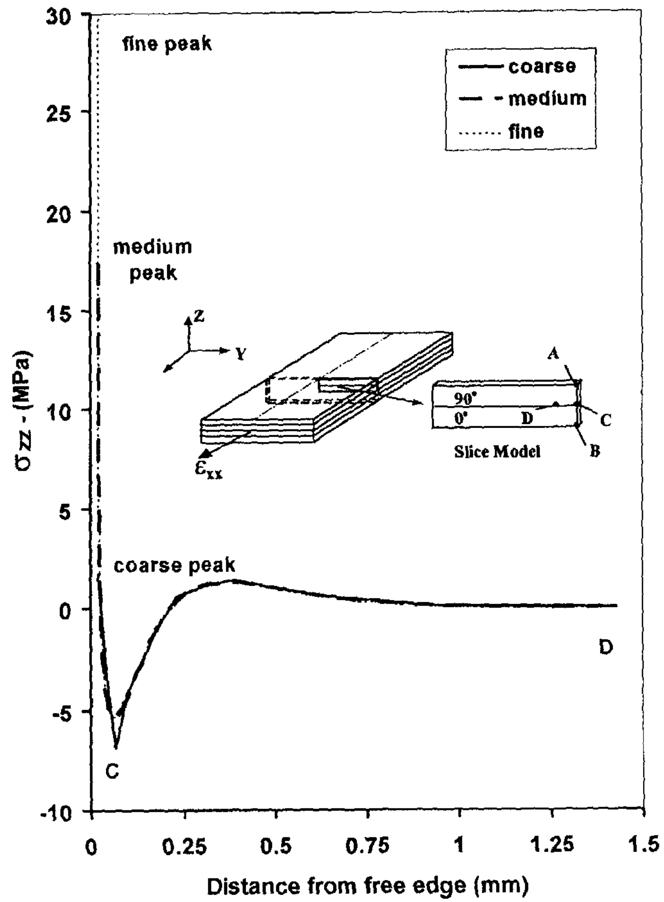

Fig. 6b. Distribution of interlaminar normal stress σ_{zz} along line CD for the $[90/0]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

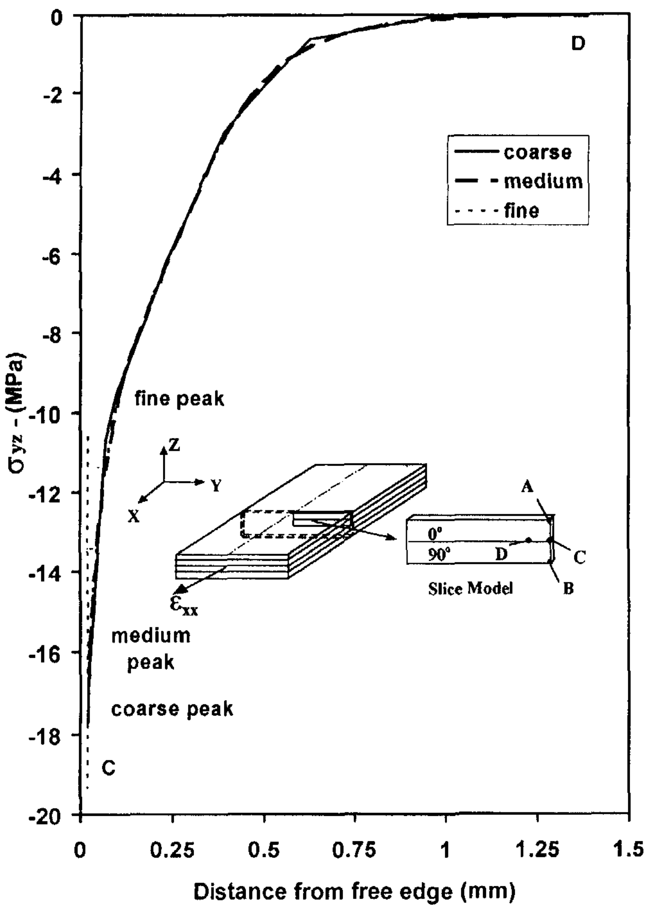

Fig. 6c. Distribution of interlaminar shear stress σ_{yz} along line CD for the $[0/90]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

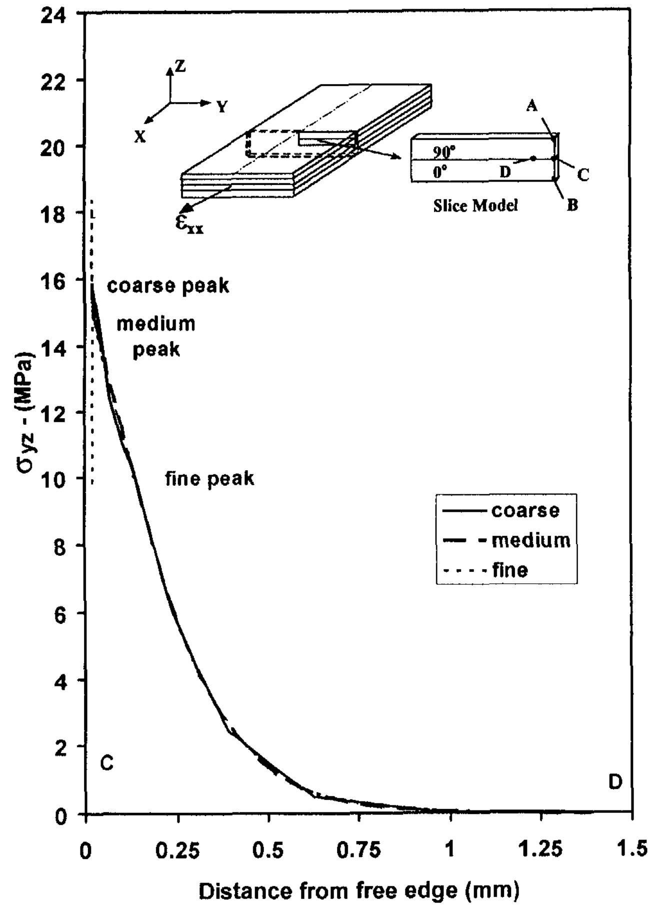

Fig. 6d. Distribution of interlaminar shear stress σ_{yz} along line CD for the $[90/0]_s$ simple composite plate. Results of fine, medium, and coarse meshing of the slice model have been plotted.

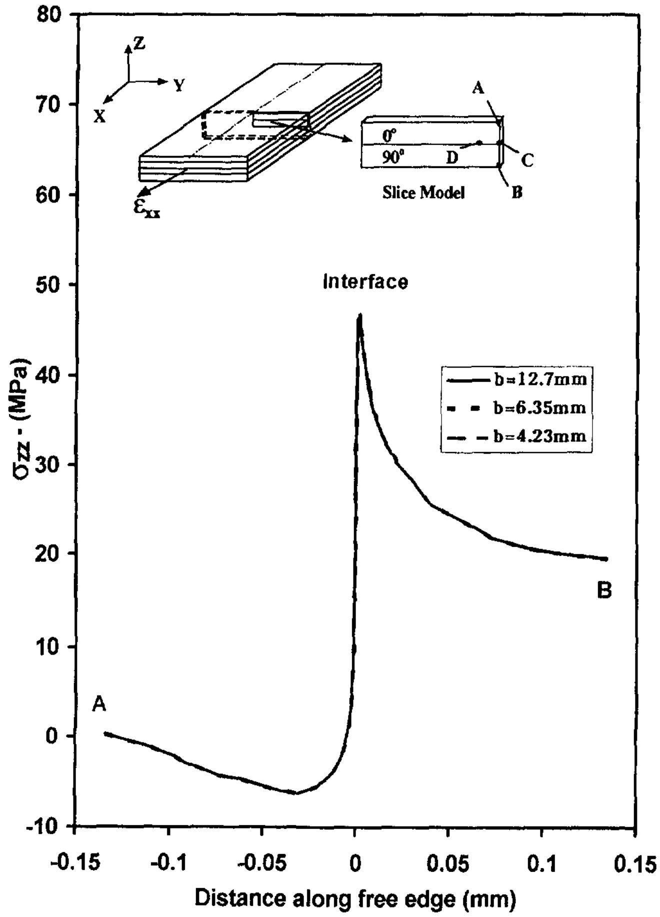

Fig. 7. Distribution of interlaminar normal stress σ_{zz} along line AB for the $[0/90]_s$ simple composite plate. Comparison of results for varying half-width, b, with constant thickness, h, of the slice model: (1) $b = 12.7$ mm, (2) $b = 6.35$ mm, and (3) $b = 4.23$ mm.

4.3. *New model results*

Another important consideration in the stress analysis of the simple composite plate is the effect of the width and the thickness on the magnitude of stresses at the free-edge. Initially, the half-width (b) of the $[0/90]_s$ medium model was varied: (1) $b = 12.7$ mm, (2) $b = 6.35$ mm, and (3) $b = 4.23$ mm. In Fig. 7, the results for σ_{zz} along the free-edge are sketched. The graph shows that the results are identical for σ_{zz} along the free-edge in all three cases. It can be concluded that for large widths, the magnitude and shape of the high stress regions are independent of the width, b. It is however known, from Pagano (1974), that for very small widths, the strength of the singularity alters for varying widths, as width approaches $b < 2h$. The exact minimum value for the onset of width independence has not been evaluated at this time.

The dependence of the magnitude of stress at the free edge with respect to the ply thickness was subsequently investigated. The ply thickness was varied by adding extra plies while width b was kept constant at 12.7 mm. In Fig. 8(a), the results of the interlaminar normal shear stress were plotted along the free edge of the specimen for the three cases: (1) $[0/90]_s$, (2) $[0_2/90_2]_s$ and (3) $[0_3/90_3]_s$. For all three models, the elements at the high stress regions (near the free edge and between the plies), are identical. It can be seen that the magnitude of interlaminar normal stress at the free edge changes for the different thicknesses. The largest magnitude of stress occurs for the $[0_3/90_3]_s$ cross-ply laminate as $\sigma_{zz} = 54.01$ MPa. As the thicknesses reduce, the magnitude of stress at the free edge also diminishes, $\sigma_{zz} = 51.35$ MPa for $[0_2/90_2]_s$ and $\sigma_{zz} = 46.85$ MPa for $[0/90]_s$. It is therefore important to conclude that the magnitude of stress at the free edge is dependent upon the ply thickness.

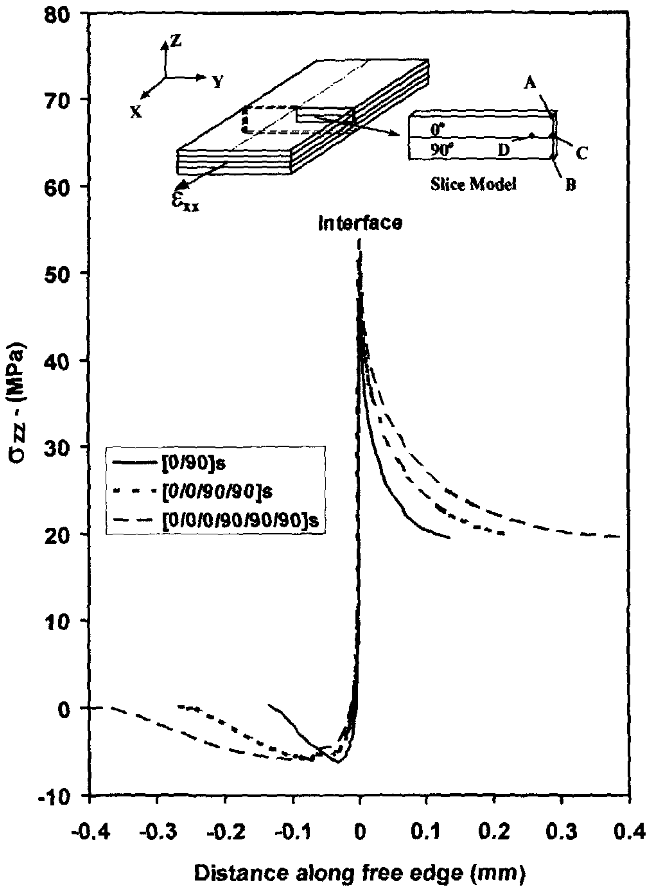

Fig. 8a. Distribution of interlaminar normal stress σ_{zz} along line AB for the $[0/90]_s$ simple composite plate. Comparison of results for varying thickness, h, with constant half-width, b, of the slice model: (1) $[0/90]_s$, (2) $[(0/90)_2]_s$ and (3) $[(0/90)_3]_s$.

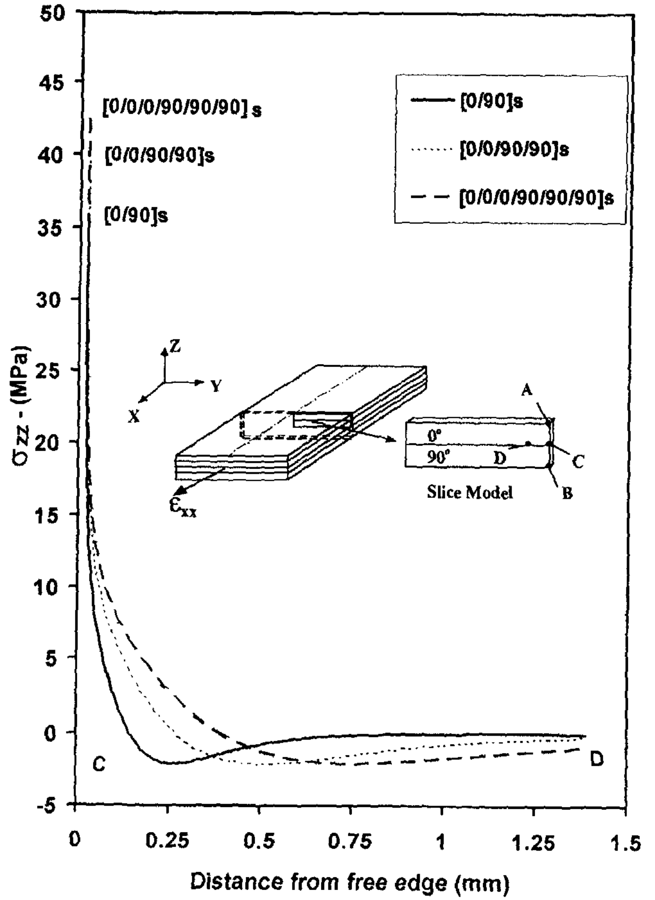

Fig. 8b. Distribution of interlaminar normal stress σ_{zz} along line CD for the $[0/90]_s$ simple composite plate. Comparison of results for varying thickness, h, with constant half-width, b, of the slice model: (1) $[0/90]_s$, (2) $[(0/90)_2]_s$ and (3) $[(0/90)_3]_s$.

The interlaminar normal stresses for the same three models have been plotted along the ply interface in Fig. 8(b). Once again the magnitude of stress at the free edge increases as the ply thickness increases. It is however interesting to note that the high stress occurs over a larger area as the ply thickness increases. These conclusions are concurrent with the work of Lagace *et al.* (1987). They found that for a $[+15/-15/0]_s$ composite plate, the area over which the stresses are high increases with increasing ply thickness at the $+15°/-15°$ ply interface. They however do not detect any change in magnitude of the stress at the free edge.

The dependence of the interlaminar stresses on b/h near the boundary layer region was then studied. Three curves were compared using the fine meshing slice model: (1) $[0/90]_s$ with $b = 12.7$ mm, (2) $[0_2/90_2]_s$ with $b = 8.26$ mm and (3) $[0_3/90_3]_s$ with $b = 4.23$ mm. Figure 9 clearly indicates that the magnitude of stress is a maximum for the $[0_3/90_3]_s$ laminate and a minimum for the $[0/90]_s$ composite plate. Once again the magnitudes of the free edge stresses are identical to those of Figs 8(a,b), $\sigma_{zz} = 54.01$ MPa for $[0_3/90_3]_s$, $\sigma_{zz} = 51.35$ MPa for $[0_2/90_2]_s$ and $\sigma_{zz} = 46.85$ MPa for $[0/90]_s$. It is concluded that the magnitude of stress is dependent on the factor b/h. This is contrary to the conclusions reached by Pagano (1974). Their results show that: "the distribution of the interlaminar stresses in the boundary layer regions is essentially independent of b/h, provided that $b > 2h$."

The results indicate that the ply thickness and/or the number of plies, h, has an effect on the magnitude of free edge stress, while there is an independence of the width, b, with respect to the magnitude of free edge stress if b is much greater than $2h$. Further results

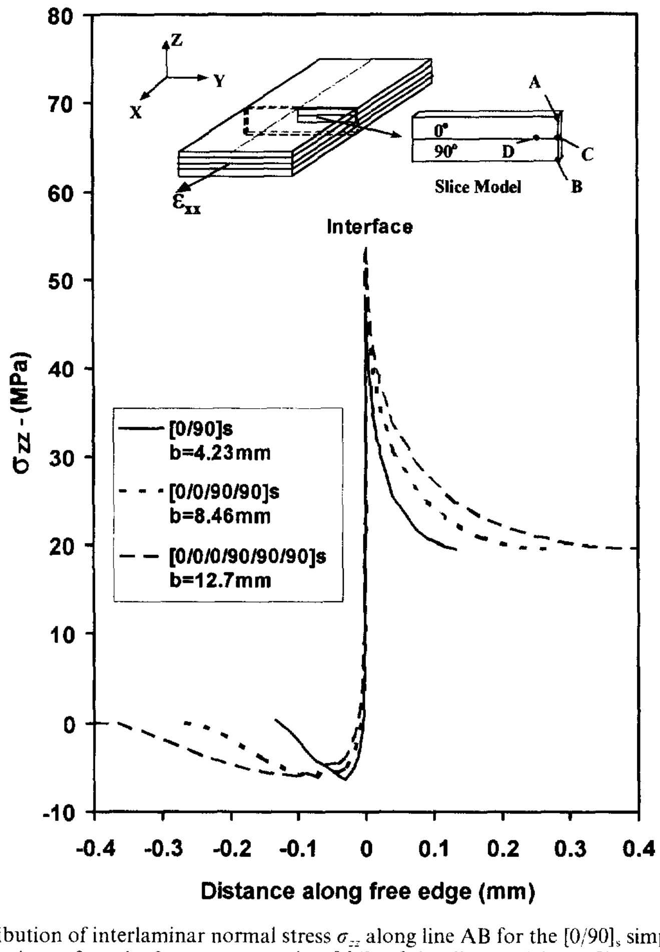

Fig. 9. Distribution of interlaminar normal stress σ_{zz} along line AB for the $[0/90]_s$ simple composite plate. Comparison of results for constant ratio of b/h, of the slice model: (1) $[0/90]_s$ with $b = 4.23$ mm, (2) $[(0/90)_2]_s$ with $b = 6.35$ mm, and (3) $[(0/90)_3]_s$, with $b = 12.7$ mm.

show that the magnitude of free edge stress is therefore also dependent upon the ratio b/h even for $b \gg 2h$ (expected results since the stresses are dependent on h and independent of b).

5. CONCLUSIONS

The present study shows the stress analysis of a simple composite plate with the use of a new technique, the "slice model". The slice model is used to represent the stress state of a simple composite plate subjected to uniaxial tension. The results of the slice model are shown to compare favorably with models from other authors. A convergence study is performed to show the importance of mesh creation to find a suitable mesh refinement for the simple composite plate.

An analysis is also carried out to determine the importance of thickness, h, and width, b, on the magnitude of stresses. For the cases examined, the results have shown that the magnitude of stress at the free edge is independent of b (for large widths), but dependent on h. As the thickness, h, increases, the magnitude of stress increases.

The important advantages of the slice model are evident when compared to other methods utilizing three-dimensional finite element analysis that have studied the problem of three-dimensional edge effects. The slice model minimizes the number of elements far from the aniticipated singularity and allows for a very fine mesh area near the critical high stress gradient regions of a composite laminate. The computer runtime is drastically reduced, while leaving sufficient computer memory to allow for an accurate stress analysis.

The slice model provides an important tool for future work. A failure analysis of the simple composite laminate is being developed by the authors with property degradation features after initial failure occurs. The slice model also allows for a higher number of elements near the anticipated stress singularity regions, therefore providing higher stress magnitudes at the edge.

Acknowledgements—The support of the Structures and Materials Laboratory of the National Research Council of Canada through a grant (grant no. 31946–1–0008/01), is gratefully acknowledged.

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